(6 Pages) **Reg. No. : ....**

Code No. : 6321 **Sub. Code : PMAE 31**

M.Sc. (CBCS) DEGREE EXAMINATION,
NOVEMBER 2021

Third Semester

Mathematics

Elective — ALGEBRAIC NUMBER THEORY

(For those who joined in July 2017 onwards)

Time : Three hours Maximum : 75 marks

PART A — (10 × 1 = 10 marks)

Answer ALL questions.

Choose the correct answers :

1. Diophantus is a ———— mathematician.

   (a) Greek (b) Latin

   (c) English (d) German

2. If $a = b =$ ———— and $c \neq 0$ then $ax + by = c$ has not solution.

   (a) 1 (b) 0

   (c) –1 (d) 2

3. If U is unimodular then $\det(U) =$ ________

(a) 1 (b) -1

(c) ± 1 (d) 0

4. IF U^{-1} exists where U is unimodular which has ________

(a) Integral (b) Zero element

(c) Elements (d) Unity

5. ________ is a zero of the form $x^2 + y^2 = z^2$.

(a) $\left(\frac{3}{5}, \frac{4}{5}, 1\right)$ (b) $\left(\frac{3}{4}, 1, 1\right)$

(c) (0, 1, 1) (d) (0, 0, 0)

6. The value of any infinite simple continued fraction $(a_0, a_1, a_2, \ldots)$ is ________

(a) zero (b) rational

(c) irrational (d) unity

7. Any divisor of the integer 1 is called a ________ of F.

(a) zero (b) rational

(c) irrational (d) unity

8. The integers of any algebraic number field from a ——————

(a) ring (b) field

(c) subfield (d) unit

9. In a quadratic field, $Q(\xi)$ where ξ is a root of an irreducible ———— polynomial over Q.

(a) Quadratic (b) Cubic

(c) Linear (d) Zero

10. The ———— of an algebraic number field form a multiplicative group.

(a) zeros (b) units

(c) roots (d) elements

PART B — (5 × 5 = 25 marks)

Answer ALL questions, choosing either (a) or (b)

11. (a) When does $ax+by=c$ have infinitely many solutions?

Or

(b) Find all integers x and y such that $147x+258y=369$.

12. (a) If u and v are relatively prime positive integers whose product uv is a perfect square, then prove that u and v are both perfect square.

Or

(b) Prove that the Diophantine equation $x^4 + x^3 + x^2 + x + 1 = y^2$ has the integral solutions (–1, 1), (0, 1), (3, 11) and not others.

13. (a) For any positive real number x, prove that $(a_0, a_1, \ldots, a_{n-1}, x) = \dfrac{xh_{n-1} + h_{n-2}}{xh_{n-1} + k_{n-2}}$.

Or

(b) Expand $\sqrt{5}$ as an infinite simple continued fraction.

14. (a) For any $n \geq 0$, prove that $\left|\xi - \dfrac{h_n}{k_n}\right| < \dfrac{1}{k_n k_{n+1}}$.

Or

(b) Let ξ denote any irrational number. If there is a rational number $\dfrac{a}{b}$ with $b \geq 1$ $\left|\xi - \dfrac{a}{b}\right| < \dfrac{1}{2b^2}$ then prove that $\dfrac{a}{b}$ equals one of the convergents of the simple continued fraction expansion of ξ.

15. (a) Prove that if m and n are two different square – free rational integers with $m, n \neq 1$ then prove that $\sqrt{m} = a + b\sqrt{n}$.

[P.T.O]

Or

(b) Prove that there are infinitely many units in any real quadratic field.

PART C — ($5 \times 8 = 40$ marks)

Answer ALL questions, choosing either (a) or (b)

16. (a) Find all solutions of $999x - 49y = 5000$.

Or

(b) Find all solutions in integers of $2x + 3y + 4z = 5$.

17. (a) Prove that the positive primitive solutions of $x^2 + y^2 = z^2$ with y even are $x = r^2 - s^2$, $y = 2rs$, $z = r^2 + s^2$, where r and s are arbitrary.

Or

(b) Prove that the equation $15x^2 - 7y^2 = 9$ has no solution in integers.

18. (a) Prove that two distinct infinite simple continued fractions converge to different values.

Or

(b) Expand $\sqrt{2}-1$ as an infinite simple continued fraction.

19. (a) If $\frac{a}{b}$ is a rational number with positive denominator such that $\left|\xi - \frac{a}{b}\right| < \left|\xi - \frac{h_n}{k_n}\right|$ for some $n \geq 1$, then prove that $b > k_n$.

Or

(b) Prove that if α is any algebraic number, then there is a rational integer b such that $b\alpha$ is an algebraic integer.

20. (a) If γ is an integer in $\mathbf{Q}(\sqrt{m})$, then prove that $N(\gamma) = \pm 1$ iff γ is a unit.

Or

(b) Let m be a negative square – free rational integer. Then prove that the field $\mathbf{Q}(\sqrt{m})$ has units ± 1.